J. Fluid Mech. (1965), *vol.* 22, *part* 3, *pp.* 561–578
Printed in Great Britain

The distortion of a magnetic field by the flow of a conducting fluid past a circular cylinder

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(Received 23 November 1964)

The distortion of a uniform magnetic field, aligned with the flow at infinity, by the potential flow of an inviscid conductor about a circular cylinder is determined. Potential flow of the fluid occurs when the interaction parameter is small; this is the case studied here. In the flow-potential and stream-function plane the problem may be formulated as a singular integral equation. Solutions of this equation show that for small fluid conductivities the magnetic field lines are distorted in the sense of being dragged along by the motion of the fluid. This process continues as the conductivity increases, with fewer and fewer of the magnetic field lines entering the body. For large conductivity this reduced flux of field lines enters over most of the body surface and exits in the neighbourhood of the rear stagnation point; behind the body there is a jet-like structure of magnetic field lines.

1. Introduction

When an inviscid, incompressible, and electrically conducting fluid flows past a body in the presence of a magnetic field, the only modification of the flow equations is the inclusion of the electromotive body force in the momentum equation. This force is proportional to the product of the current generated by the distortion of the magnetic field with the square of the Alfvén number, i.e. the ratio of the Alfvén speed to the flow speed. Since the current generated by this distortion is proportional to the magnetic Reynolds number, the additional body-force term, the Lorentz force, is proportional to an interaction parameter which is the product of the square of the Alfvén number with the magnetic Reynolds number. In dimensional quantities this parameter is essentially the product of the square of the magnetic field strength with the conductivity of the fluid.

This paper treats the flow of such a fluid when this interaction parameter is small. We consider flow past a two-dimensional body that is symmetrical about the uniform flow direction in the presence of a magnetic field aligned with the flow at infinity. Since the interaction parameter is small, the flow field is considered to be the usual potential flow. This flow, however, causes a modification in the magnetic field that is the order of the magnetic Reynolds number or one, whichever is smaller. The appropriate Green's function is used to formulate a singular integral equation for the distribution of the magnetic field strength on the surface of a circular cylinder. The solution of this equation then determines the magnetic field throughout the flow field as well as inside the body. Tamada (1961*a*, *b*, 1964) and Leonard (1962) have considered such flows by quite different methods. For example, in the case of a highly conducting fluid in the presence of a weak magnetic field, Tamada has found the approximate distribution of the magnetic field through recourse to the Sears (1961) boundary-layer approximation and a similarity solution.† This boundary-layer solution exhibits a singular behaviour at the trailing edge which is attributable to a failure of the boundary-layer approximation. Such singular behaviour is removed here by our more rigorous treatment. Tamada has also found the solution for the flow of a slightly conducting fluid past a circular cylinder under Oseen's approximation. In this case the exact result obtained in this paper differs only in the higher-order terms from the earlier result. Leonard has constructed solutions for a circular cylinder in a channel by relaxation techniques. His results are qualitatively similar to those obtained here. He also carries out the first iteration on the velocity field by the same technique.

2. Formulation

For the two-dimensional motion of an incompressible conducting fluid the pertinent equations in non-dimensional form are

$$\nabla.\mathbf{q} = 0, \tag{1}$$

$$\mathbf{q}.\nabla(\nabla\times\mathbf{q}) = \frac{\mu H_\infty^2}{4\pi\rho U^2}\mathbf{H}.\nabla(\nabla\times\mathbf{H}) = A^2\mathbf{H}.\nabla(\nabla\times\mathbf{H}), \tag{2}$$

$$\nabla.\mathbf{H} = 0, \tag{3}$$

and

$$\nabla\times\mathbf{H} = \sigma UL(\mathbf{q}\times\mathbf{H}) = R_m(\mathbf{q}\times\mathbf{H}). \tag{4}$$

Here $\mathbf{q}$ and $\mathbf{H}$ are the velocity and magnetic field vectors non-dimensionalized by the magnitude of their values $\mathbf{U}$ and $\mathbf{H}_\infty(\mathbf{U}||\mathbf{H}_\infty)$ at infinity; ρ, μ, and σ are the density, magnetic permeability, and electrical conductivity, and L is a typical body dimension. The magnetic Reynolds number R_m and the Alfvén number A are defined by the above equations. Combining the curl of the momentum equation (2) with Ohm's law (4) we see that, when the interaction parameter A^2R_m is small, we may neglect the effect of the magnetic field on the flow and take the velocity field to be simply that of the potential flow $\mathbf{q}_0$ past the body. With this simplification and upon introducing the magnetic stream function Ψ, we find from equation (4) that

$$\nabla^2\Psi - R_m\mathbf{q}_0.\nabla\Psi = 0 \tag{5}$$

outside the body. Inside the body, when there is no applied electric field, Ψ is harmonic. Thus our problem is to find a solution of equation (5) outside the body that is harmonic inside the body and satisfies the condition that $\mathbf{H}$ must be continuous across the surface of the body.

† A referee has called the authors' attention to the paper by Hocking (1961) in which a similar problem has been treated. Hocking considered the flow of a highly conducting fluid past a cylinder weakly magnetized as a dipole.

The dependence of equation (5) on the specific body, that is on $\mathbf{q}_0$, may be removed by introducing as independent variables the potential x and stream function y for the flow $\mathbf{q}_0$. These peculiar designations have been chosen because we shall be mainly concerned with the (x, y)-plane. In terms of our new variables, (5) becomes Oseen's equation

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-R_m\frac{\partial\Psi}{\partial x}=0.$$

The Jacobian of the mapping vanishes only at stagnation points. Under this mapping the region outside the body is mapped onto the (x,y)-plane and the body into the slit $y = 0$ with a length that is simply the change in potential from the leading to the trailing edge. We may normalize the potential so that this change is 2; thus the body is mapped into the slit $y = 0$, $|x| \leqslant 1$. The mapping is of course double-valued, with the interior of the body also being mapped onto the (x, y)-plane, the upper half of the body going into the lower half plane. The region of the lower half plane that is the map of the interior of the body depends upon the shape of the body. For a circular cylinder this region is the entire half plane, and we shall restrict our attention to this case. For other shapes, our method of solution is still applicable in principle; however, for any given body the actual details depend critically upon this mapping.

Because the problem is symmetrical about the axis of the body we consider the mapping of the upper half of the physical plane onto the full (x,y)-plane: the upper half of the flow field is mapped onto the upper half plane, the upper half of the body onto the lower half plane. If we let $\Psi^{\pm}$ be the values of Ψ in the upper and lower half planes, then our problem may be reformulated as

$$\left(\nabla^2-R_m\frac{\partial}{\partial x}\right)\Psi^+=0,\quad (y>0);$$

$$\nabla^2\Psi^-=0,\quad (y<0).$$

The condition of continuity of the magnetic field across the surface of the body in the transformed plane is simply that Ψ and its derivative with respect to y be continuous across the slit.

We propose to formulate the problem as an integral equation via the appropriate Green's functions. To do this we introduce the magnetic stream functions $\psi^{\pm}: \Psi^+ = y+\psi^+$, $\Psi^- = \psi^-$. Note that ψ is not the perturbed magnetic stream function in the physical plane; it does, however, vanish in both planes at large distances from the body. Thus we may formulate the problem as follows:

$$\frac{\partial^2\psi^+}{\partial x^2}+\frac{\partial^2\psi^+}{\partial y^2}-R_m\frac{\partial\psi^+}{\partial x}=0,\quad (y>0),$$

$$\frac{\partial^2\psi^-}{\partial x^2}+\frac{\partial^2\psi^-}{\partial y^2}=0,\quad (y<0).$$

These equations are subject to the boundary conditions

$$\psi^{\pm}\to 0\quad \text{as}\quad x^2+y^2\to\infty,$$

$$\psi^+=\psi^-=0\quad \text{on}\quad y=0,\quad |x|\geqslant 1,$$

$$\psi^+=\psi^-,\quad \frac{\partial\psi^-}{\partial y}-\frac{\partial\psi^+}{\partial y}=1\quad \text{on}\quad y=0,\quad |x|<1.$$

By transforming the dependent variable in Oseen's equation we can obtain Helmholtz's equation. Under the prescribed boundary conditions the appropriate singular solution is the zeroth-order modified Bessel function of the second kind. Applying Green's formula we may then deduce the appropriate Green's function. The results are

$$\psi^{+} = \frac{k}{\pi} y e^{kx} \int_{-1}^{1} \frac{f(\xi)\, e^{-k\xi} K_1[k\sqrt{\{(x-\xi)^2+y^2\}}]}{\sqrt{\{(x-\xi)^2+y^2\}}} d\xi, \tag{6}$$

and of course for $k = 0$,

$$\psi^{-} = -\frac{y}{\pi} \int_{-1}^{1} \frac{f(\xi)}{(x-\xi)^2+y^2} d\xi, \tag{7}$$

where $k = \frac{1}{2}R_m$, ξ is a dummy variable, $f(\xi)$ is the unknown value of $\psi(\xi, 0)$ for $|\xi| < 1$, and K_1 is the modified Bessel function of the first order. The boundary condition

$$\left.\frac{\partial\psi^{-}}{\partial y}\right|_{y=0} - \left.\frac{\partial\psi^{+}}{\partial y}\right|_{y=0} = 1 \quad (|x| < 1), \tag{8}$$

results in an integral equation for $f'(\xi)$. In their present form the derivatives of equations (6) and (7) with respect to y do not exist. We may overcome this difficulty simply by integrating both equations by parts. Because $\psi^{\pm}(\pm 1, 0) = 0$ only the integrals remain. We may now perform the required differentiation and take the limit of the resulting expressions as y tends towards zero; applying the boundary condition (8) we find

$$\frac{1}{\pi}\int_{-1}^{1} f'(\xi)\,[1 + e^{k(x-\xi)} k|x-\xi|\, K_1(k|x-\xi|)] \frac{d\xi}{x-\xi} \\ + \frac{k^2}{\pi}\int_{-1}^{1} f(\xi) \left\{ e^{k(x-\xi)} \left[K_0(k|x-\xi|) - \frac{|x-\xi|}{x-\xi} K_1(k|x-\xi|) \right] \right\} d\xi = 1.$$

Observing that the terms in the curly bracket are a perfect differential, we obtain the equation for $f'(\xi)$ by integrating by parts. Quadrature of the solution to the resulting singular integral equation,

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f'(\xi)}{x-\xi} \{1 + e^{k(x-\xi)}[k|x-\xi|\, K_1(k|x-\xi|) + k(x-\xi)\, K_0(k|x-\xi|)]\} d\xi = 1 \tag{9}$$

and the formulas (6) and (7) provide the solution for the magnetic stream function, both inside and outside the body.

The integral equation (9) appears insoluble in full because of the complicated nature of the kernel. In the next two sections we obtain the appropriate analytical solutions for small and large values of magnetic Reynolds number. We conclude with a numerical solution, valid for small and moderate values of the magnetic Reynolds number, that substantiates our analytical solutions.

3. Small magnetic Reynolds number

When the magnetic Reynolds number is small our requirement that the interaction parameter be small is automatically satisfied for moderate values of the magnetic field strength. We separate the equation (9) into its singular and non-singular parts, and introduce the function

$$F(x) = 1 - \frac{1}{\pi}\int_{-1}^{1} f'(\xi) \left[\frac{e^{k(x-\xi)} k|x-\xi|\, K_1(k|x-\xi|) - 1}{x-\xi} + e^{k(x-\xi)} k K_0(k|x-\xi|) \right] d\xi.$$

Upon inverting the singular part of the equation there results

$$f'(\xi) = \frac{1}{2\pi\sqrt{(1-\xi^2)}}\int_{-1}^{1}\frac{F(x)\sqrt{(1-x^2)}}{x-\xi}dx + \frac{\Gamma}{\sqrt{(1-\xi^2)}}, \tag{10}$$

where Γ is a constant that is determined by the requirement $f(1) = 0$. Quadrature of equation (10) shows that $\Gamma = 0$. When this equation for $f'(\xi)$ is substituted into the definition for $F(x)$ and a permissible change is made in the order of integration, we obtain a Fredholm equation of the first kind,

$$F(x) + \frac{1}{2\pi^2}\int_{-1}^{1} F(\eta)\sqrt{(1-\eta^2)}\, I(k;\, \eta, x)\, d\eta = 1, \tag{11}$$

where I is the function

$$I(k;\, \eta, x) = \int_{-1}^{1}\frac{1}{\sqrt{(1-\xi^2)}}\left[\frac{k e^{k(x-\xi)}|x-\xi|\, K_1(k|x-\xi|)-1}{x-\xi} + k e^{k(x-\xi)}K_0(k|x-\xi|)\right]\frac{d\xi}{\eta-\xi}. \tag{12}$$

To obtain a solution for small k we expand the integrand of I for small $k(x-\xi)$. The resulting integrals may then be evaluated by contour integration. Integration is facilitated by introducing the angular co-ordinates:

$$x = -\cos\alpha, \quad \xi = -\cos\theta, \quad \eta = -\cos\beta.$$

In this notation equations (10), (11), and (12) become

$$f'(\theta) = \frac{1}{2\pi}\int_0^{\pi}\frac{F(\beta)\sin^2\beta}{\cos\theta-\cos\beta}d\beta, \tag{13}$$

$$F(\alpha) = 1 - \frac{1}{2\pi^2}\int_0^{\pi} F(\beta)\sin^2\beta\, \tilde{I}(k;\, \beta, \alpha)\, d\beta, \tag{14}$$

$$\tilde{I}(k;\beta,\alpha) = I(k; -\cos\beta, -\cos\alpha). \tag{15}$$

For small k we find

$$\tilde{I}(k;\beta,\alpha) = \tfrac{1}{2}\pi k^2(\tfrac{1}{2}-\gamma-\log\tfrac{1}{4}k) - k(1-\tfrac{1}{2}k\cos\alpha)\, I_0^* - \tfrac{1}{2}k^2 I_1^* + O(k^3\log k),$$

where

$$I_n^*(\alpha,\beta) = \int_0^{\pi}\frac{\cos n\theta \log 2|\cos\theta-\cos\alpha|}{\cos\theta-\cos\beta}d\theta$$
$$= -\frac{\pi}{\sin\beta}\sum_{m=1}^{\infty}\frac{\cos m\alpha}{m}[\sin(m+n)\beta + \sin(m-n)\beta],$$

γ is Euler's constant, and m and n are integers. Note that the partial derivative of I_0^* with respect to α in integral form vanishes, but that the same operation must not be applied to the sum. This is clear from the graph of I_0^* as a function of α for fixed β. The function I_0^* is the step function that jumps from a value of $\pi(\beta-\pi)(\sin\beta)^{-1}$ for $\alpha < \beta$ to $\pi\beta(\sin\beta)^{-1}$ for $\alpha > \beta$; the derivative is everywhere zero except at β.

Substituting the integral $\bar{I}$ into (14) we then obtain

$$F(\alpha) = 1+\frac{k^2}{4\pi}\left(\frac{2\gamma-1}{2}+\log\tfrac{1}{4}k\right)c_1-\frac{k}{\pi}(1-\tfrac{1}{2}k\cos\alpha)\sum_{m=1}^{\infty}\frac{c_m\cos m\alpha}{m}$$
$$-\frac{k^2}{4\pi}\sum_{m=1}^{\infty}(c_{m+1}+c_{m-1})\frac{\cos m\alpha}{m}+O(k^3\log k),$$

where $$c_n = \int_0^\pi F(\beta)\sin\beta\sin n\beta\,d\beta \quad (n = 0,1,2\,\ldots).$$

Multiplying both sides of the above equation by $\sin\alpha\sin n\alpha$ and integrating from 0 to π we find

$$c_1 = \tfrac{1}{2}\pi\left[1+\frac{c_1k^2}{4\pi}(\gamma+\log\tfrac{1}{4}k)+\frac{kc_2}{4\pi}+\frac{k^2}{16\pi}(c_1+\tfrac{1}{3}c_3)\right], \tag{16}$$

and for $n > 1$

$$c_n = -\tfrac{1}{4}k\left[\frac{c_{n-1}}{n-1}-\frac{c_{n+1}}{n+1}-\tfrac{1}{4}k\left(\frac{c_{n-2}}{(n-1)(n-2)}-\frac{2c_n}{n^2-1}+\frac{c_{n+2}}{(n+1)(n+2)}\right)\right]. \tag{17}$$

Clearly $c_n = O(k^{n-1})$, and we may replace the c_n's on the right-hand sides of equations (16) and (17) by lower-order approximations to compute the c_n's correctly to any given order. For example, in (16) we may replace c_1 by $\frac{1}{2}\pi$ on the right-hand side with an error of $O(k^4\log k)$ in c_1. Thus we may compute the required c_n's to find

$$F(\alpha) = 1-\tfrac{1}{2}k\cos\alpha+\tfrac{1}{8}k^2\log\tfrac{1}{4}k+\tfrac{1}{16}k^2[2\gamma+1+2\cos 2\alpha]+O(k^3\log k).$$

We are now in a position to compute $f'(\theta)$ by using equation (13); the required integrals may be evaluated by contour integration around the unit circle. Quadrature with respect to θ yields the solution

$$f(\theta) = \frac{1}{2}\left[1-\frac{k}{4}\cos\theta+\frac{k^2}{8}\log\frac{k}{4}+\frac{k^2}{12}\left(\frac{6\gamma-1}{4}+\cos^2\theta\right)\right]\sin\theta+O(k^3\log k). \tag{18}$$

Consider the case of a circular cylinder with radius 1, a flow with $U = 1$, and a magnetic field with $H_\infty = 1$. Under our mapping to the (x,y)-plane the cylinder is mapped into a slit of length 4. Thus the distribution of the magnetic field on the surface of the cylinder may be obtained by multiplying equation (18) by two and replacing k by R_m; this effectively shrinks the slit length to 2. The angular variable θ is to be interpreted as the polar angle measured from the forward stagnation point. The result (18) is in agreement with the earlier result of Tamada (1961*a*) to $O(R_m^2\log R_m)$; a discrepancy in the term $O(R_m^2)$ is attributed to the Oseen approximation made in the earlier paper. The change in the total magnetic flux entering the cylinder is

$$\frac{R_m^2}{8}\log\frac{R_m}{4}+\frac{R_m^2}{144}(18\gamma+19).$$

For $R_m = \frac{1}{2}$ the flux entering the body is decreased by approximately $4\frac{1}{2}\%$

The magnetic-field distribution inside the body, which may be computed from equation (7), or more conveniently in this example from Poisson's formula, is given by

$$\Psi(r,\theta) = r\sin\theta\left[1+\frac{R_m^2}{8}\log\frac{R_m}{4}+\frac{\gamma R_m^2}{8}\right]-\frac{R_m}{8}r^2\sin 2\theta$$
$$+\frac{R_m^2}{48}r^3\sin 3\theta+O(R_m^3\log R_m).$$

Outside the cylinder the magnetic stream function may be computed by using equation (6) and transforming the results to the physical plane. Figure 1 is a sketch of the magnetic-field distribution for $R_m = \frac{1}{2}$.

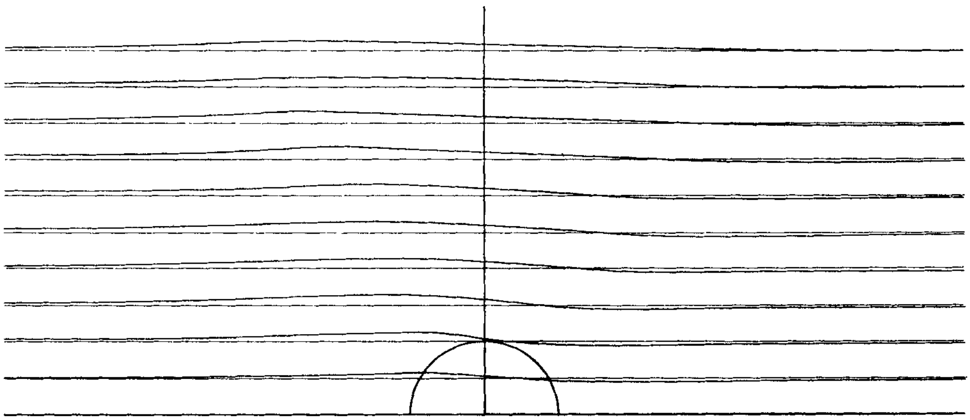

FIGURE 1. Magnetic field for small R_m.

4. Large magnetic Reynolds number

As the magnetic Reynolds number increases, more and more of the magnetic field will be convected with the fluid, and, in a sense, dragged out of the body by the motion of the highly conducting fluid about the body. Indeed, for large R_m the total magnetic field inside the body is $O(R_m^{-\frac{1}{2}})$ of that which would pass through the body if the fluid were non-conducting. With the magnetic Reynolds number large it is necessary that the magnetic-field strength be very small, $A_\infty^2 = o(R_m^{-1})$, for the interaction parameter to be considered negligible throughout the flow.

Here it is more expedient to work with the singular form of the integral equation, which we rewrite for convenience as

$$\frac{1}{\pi}\int_{-1}^{1} f'(\xi)\,\mathscr{K}(x-\xi)\,d\xi = 1, \tag{19}$$

where
$$\mathscr{K}(u) = u^{-1}+k\,e^{ku}[\operatorname{sgn}(u)\,K_1(k|u|)+K_0(k|u|)]. \tag{20}$$

Apart from a few special kernels, equations of the form (19) cannot be inverted by techniques now available. However, if the interval of definition in equation (19) were semi-infinite, then the equation would be amenable to treatment by the Wiener–Hopf technique. If we remove the normalization of the change in velocity potential introduced earlier, and take this change to be L (L is the integral of the velocity on the body over the upper or lower surface), and if we change variables from (x, ξ) to $y = \frac{1}{2}Lk(x+1)$, $\eta = \frac{1}{2}Lk(\xi+1)$, then (19) becomes

$$\frac{1}{\pi}\int_0^{kL}\frac{f'(\eta)}{y-\eta}\{1+e^{y-\eta}[|y-\eta|\,K_1(|y-\eta|)+(y-\eta)\,K_0(|y-\eta|)]\}\,d\eta = \frac{1}{k}. \tag{21}$$

With $Lk \to \infty$, but k fixed, this becomes a Wiener–Hopf problem for the magnetic field inside a semi-infinite body. If $k \to \infty$ with L fixed, the only solution is $f'(\eta) = 0$. For the semi-infinite body, the integral of the solution to equation (21) is

$$f(\eta) = \frac{1}{k}\sqrt{\frac{2\eta}{\pi}}. \tag{22}$$

This result is obtained from a straightforward application of the Wiener–Hopf technique to (21) with $kL = \infty$; the Fourier transform of the kernel can be factorized by inspection. The result (22) shows that the magnetic field continues to enter the body over its entire length. For a finite body we can expect this solution to be valid near the leading edge.

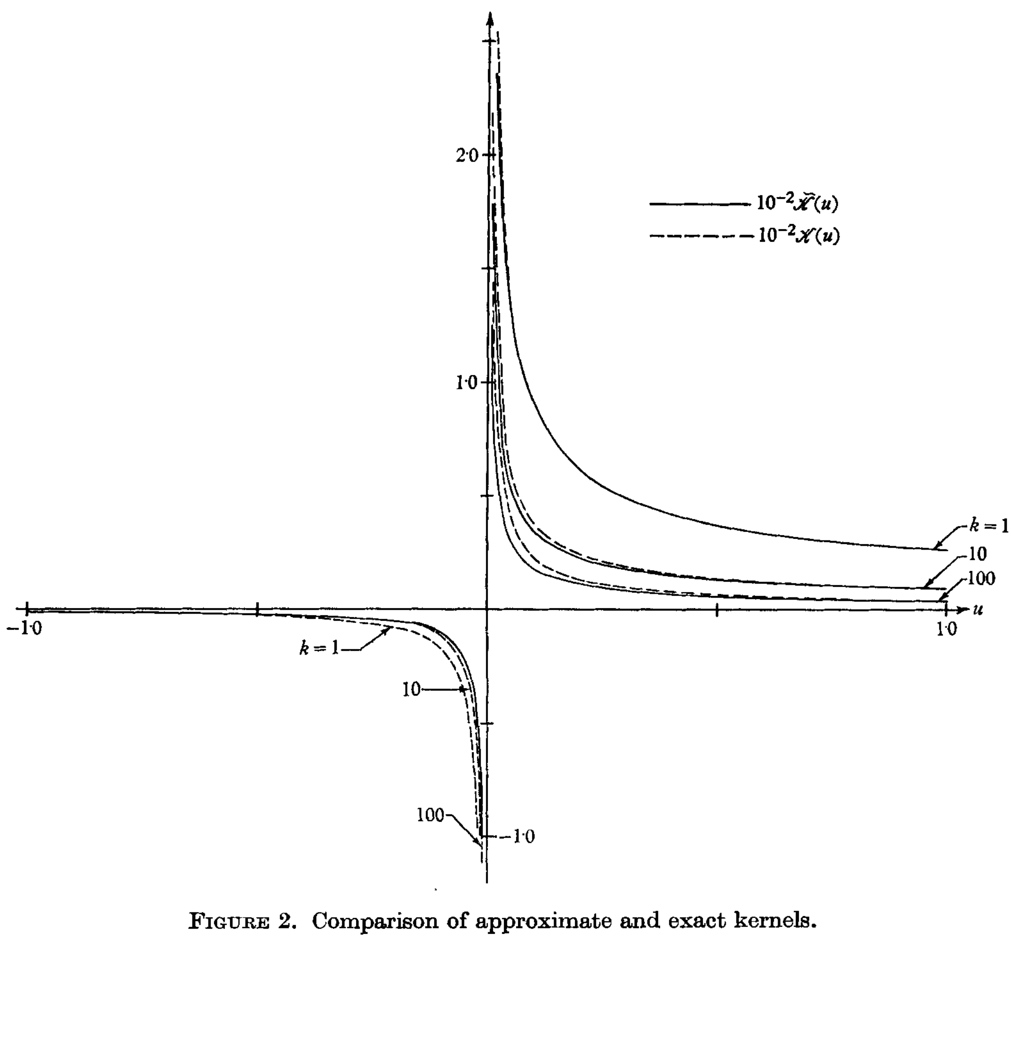

FIGURE 2. Comparison of approximate and exact kernels.

With the exception of one sign, the kernel (20) is the same as the kernel that occurs in the problem of Oseen flow past a flat plate; here in contradistinction to the Oseen problem, the singular terms add. Although there is no asymptotic expansion of the kernel for large k which is uniformly valid for all u, the first two terms in the asymptotic expansion for $\mathscr{K}(u)$ leads to a useful approximate kernel $\tilde{\mathscr{K}}(u)$

$$\tilde{\mathscr{K}}(u) = u^{-1} + \sqrt{(\pi k/2|u|)}\,[\operatorname{sgn}(u) + 1].$$

This approximate kernel is compared to the full kernel in figure 2. It has the property of being an asymptotically valid approximation to $\mathscr{K}(u)$ for all $u > \epsilon(k) > 0$. The power-series expansion of $\mathscr{K}(u)$ for small values of ku shows that $\epsilon(k) = O(k^{-1})$. Furthermore, $\tilde{\mathscr{K}}(u)$ retains the correct singular behaviour as $u \to 0$: $\tilde{\mathscr{K}}(u) \to \frac{1}{2}\mathscr{K}(u)$. A comparison of the integrals of these two kernels over the interval $[-1, 1]$ (for the necessary asymptotic expansions see Luke 1962) shows that

$$\int_{-1}^{1} \mathscr{K}(u)\,du = 2\sqrt{(2\pi k)}\left[1 + \frac{31}{4k} + O(k^{-2})\right],$$

the first term on the right-hand side being the integral of $\tilde{\mathscr{K}}(u)$. The Cauchy principal value makes the error for small u inconsequential.

The approximate kernel $\tilde{\mathscr{K}}(u)$ is correct to second order in $k^{-\frac{1}{2}}$ except in the neighbourhood of $|u| \leqslant O(k^{-1})$. If a term is included that corrects the approximate kernel so that it has the proper behaviour as $u \to 0$, then one can show that the solution is unaffected to second order. Since the approximate kernel is correct only to this order, and further terms in the asymptotic expansion make the integral divergent, there is no rationale in making such a correction.

We proceed under the assumption that for large k the replacement of $\mathscr{K}(u)$ by $\tilde{\mathscr{K}}(u)$ in equation (19) is valid to second order. This reduces the problem to the solution of the integral equation

$$\int_{-1}^{1} \frac{f'(\xi)}{x-\xi}\,d\xi + \sqrt{(2\pi k)} \int_{-1}^{x} \frac{f'(\xi)}{\sqrt{(x-\xi)}}\,d\xi = \pi. \tag{23}$$

Equation (23) is deceptively simple in appearance; both integrals can be inverted separately, and one might presume that a simple inversion exists for all k. However, difficulties are soon encountered, e.g. the related Fredholm equation has an elliptic function for its kernel. Because the equation is appropriate to the problem at hand only for large k, and even then only to second order, we use the iteration scheme $f(\xi) = f_0(\xi) + f_1(\xi) + \ldots$, where $f_0'(\xi)$ and $f_1'(\xi)$ are determined by inversion of the Abel integral equations

$$\int_{-1}^{x} \frac{f_0'(\xi)}{\sqrt{(x-\xi)}}\,d\xi = \sqrt{\left(\frac{\pi}{2k}\right)},$$

and

$$\int_{-1}^{x} \frac{f_1'(\xi)}{\sqrt{(x-\xi)}}\,d\xi = -\frac{1}{\sqrt{(2\pi k)}} \int_{-1}^{1} \frac{f_0'(\xi)}{x-\xi}\,d\xi.$$

The first two iterations can be carried out in closed form. Omitting the somewhat cumbersome details, we find the distribution of the magnetic stream-function along the slit in the transformed plane to be

$$f(\alpha) = \frac{2\sin\frac{1}{2}\alpha}{\sqrt{(\pi k)}} - \frac{\alpha}{2\pi^2 k}\left[1+\sum_{n=1}^{\infty}\frac{(1-2^{1-2n})}{(2n+1)!}B_n\alpha^{2n}\right]+O(k^{-\frac{3}{2}}), \tag{24}$$

where $x = -\cos\alpha$, and the B_n are the Bernoulli numbers.

The first term of this result is just that obtained with recourse to a similar solution of the magnetohydrodynamic-boundary-layer equations (Tamada 1961*b*, 1964), and is identical to the leading-edge solution (22). Equation (24) represents a slight improvement (when compared to numerical results) over (22). However, the solution (24) has the perverse property of not vanishing at the end of the slit. In fact,

$$f(\pi) = \frac{2}{\sqrt{(\pi k)}} - \frac{2G^*}{\pi^2 k} + O(k^{-\frac{3}{2}}),$$

where G^* is the alternating series of reciprocals of the odd numbers: $G^* \approx 0{\cdot}916$. Thus the distribution of the magnetic stream-function does not vanish at the rear stagnation point; there is a singularity in the magnetic-field strength there since, by symmetry, the zero streamline must pass through both of the stagnation points. Because of this singularity, the proposed iteration procedure fails near the trailing edge, and we must modify our solution there.

Except in the neighbourhood of the rear stagnation point, our leading-edge solution (24) is valid. To this solution, $f_L(x)$, we must match a solution $f_T(x)$ valid for $x-1 = o(1)$. To obtain the required solution,† we transform from (x,ξ) to (t,η) co-ordinates, where $k(x-1) = t$, and $k(\xi-1) = \eta$. Under this transformation equation (19) becomes

$$\frac{1}{\pi}\int_{-2k}^{0} f'(\eta)\left\{\frac{1}{t-\eta}+e^{t-\eta}\left[\operatorname{sgn}(t-\eta)\,K_1(|t-\eta|)+K_0(|t-\eta|)\right]\right\}d\eta = \frac{1}{k}.$$

If we take the limit $k\to\infty$, we then have an equation amenable to solution by the Wiener–Hopf technique

$$\frac{1}{\pi}\int_{-\infty}^{0} f'(\eta)\left\{\frac{1}{t-\eta}+e^{t-\eta}\left[\operatorname{sgn}(t-\eta)\,K_1(|t-\eta|)+K_0(|t-\eta|)\right]\right\}d\eta = 0. \tag{25}$$

To carry out the solution, we define the functions

$$G(\eta) = \begin{Bmatrix} f_T'(\eta) & \text{for} & \eta<0 \\ 0 & \text{for} & \eta>0 \end{Bmatrix}, \quad \text{and} \quad H(t) = \begin{Bmatrix} 0 & \text{for} & t<0 \\ h(t) & \text{for} & t>0 \end{Bmatrix},$$

where $h(t)$ is an unknown function, and re-write equation (25) as

$$\int_{-\infty}^{\infty}\left\{\frac{1}{t-\eta}+e^{t-\eta}\left[\operatorname{sgn}(t-\eta)\,K_1(|t-\eta|)+K_0(|t-\eta|)\right]\right\}G(\eta)\,d\eta = H(t). \tag{26}$$

† The authors are indebted to a referee for a suggestion that resulted in an improvement in our results for the trailing-edge region.

We then apply the Fourier transformation, defined here as

$$\hat{M}(p) = \int_{-\infty}^{\infty} M(t)\, e^{-ipt} dt,$$

to equation (26) to obtain

$$\pi \left\{ \frac{2-ip}{\sqrt{[p(p+2i)]}} - \lim_{\epsilon\to 0} \frac{ip}{\sqrt{(p^2+\epsilon^2)}} \right\} \hat{G}(p) = \hat{H}(p), \tag{27}$$

where ϵ is a small positive real quantity.

The function $\hat{G}(p)$ is analytic in the upper half of the p-plane, while $\hat{H}(p)$ is analytic in the lower half of the p-plane. Factorization of the function multiplying $\hat{G}(p)$ may be carried out by inspection, with the result

$$-i\pi \left\{ \sqrt{(p+2i)} + \lim_{\epsilon\to 0} \frac{p}{\sqrt{(p+i\epsilon)}} \right\} \hat{G}(p) = \lim_{\epsilon\to 0} \hat{H}(p)\, \sqrt{(p-i\epsilon)}.$$

In this expression the left-hand side represents a function which is analytic in the upper half of the p-plane; the right-hand side represents a function analytic in the lower half of the p-plane. Together both sides represent by analytic continuation the same function, which is analytic everywhere except on the real axis and at infinity. Anticipating that $G(t) \sim (-t)^{-\frac{1}{2}}$ as $t \to 0$, we require

$$\hat{G}(p) \sim p^{-\frac{1}{2}} \quad \text{as} \quad p \to \infty$$

and therefore take this function to be a constant B. The inversion of $\hat{G}(p)$ then gives

$$G(t) = \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ipt}}{\sqrt{(p)} + \sqrt{(p+2i)}}\, dp, \quad (-\tfrac{1}{2}\pi < \arg p < \tfrac{3}{2}\pi),$$

where the cut along the negative imaginary axis of the p-plane is prescribed to make the integrand single-valued. A simple change in variables then reduces this integral to the inversion of $(p)^{\frac{1}{2}}$, with the result

$$G(t) = \frac{C}{(-t)^{\frac{3}{2}}} [1 - e^{2t}],$$

where C is a constant replacing a complex number times B. Since $f_T'(t) = G(t)$ for $t < 0$, we obtain the distribution of the magnetic stream-function by quadrature of the above result and using $f_T(0) = 0$

$$f_T(t) = 2C\{(-t)^{-\frac{1}{2}}(1-e^{2t}) - \sqrt{(2\pi)}\, \operatorname{erf} \sqrt{(-2t)}\}.$$

As $t \to -\infty$, we require $f_T(t) \to f_L(1)$. This completes the solution by determining C. In terms of x, the trailing-edge solution is

$$f_T(x) = \frac{-2}{\pi\sqrt{k}} \left[1 - \frac{G^*}{\pi\sqrt{(\pi k)}} \right] \left[\frac{1-e^{-2k(1-x)}}{\sqrt{\{2k(1-x)\}}} - \sqrt{\pi}\, \operatorname{erf} \sqrt{\{2k(1-x)\}} \right]. \tag{28}$$

The two solutions $f_L(x)$ and $f_T(x)$ for various values of k are shown in figure 3. To estimate the extent of the trailing-edge region we find the value of x, for which the two solutions are equal. For large k, and hence small values of $1-x$, the result is $1 - x = 2(\pi k)^{-\frac{1}{3}}$, the correct matching coming from the condition

$(1-x)k \gg 1$. Because of the square-root singularity in the mapping back to the physical plane, the extent of this region for bodies with a blunt trailing edge is $O(R_m^{-\frac{1}{2}})$ in the physical plane.

Inside the body, but away from the rear stagnation point, the magnetic-field strength is clearly $O(R_m^{-\frac{1}{2}})$ of that at large distances from the body; the tangential and normal components of the field at the surface are also $O(R_m^{-\frac{1}{2}}H_\infty)$. However, bodies with blunt trailing edges will have a magnetic-field strength that is $O(H_\infty)$ at the trailing edge in the physical plane. This last result is a consequence of two facts: the component of the magnetic field normal to the body in the

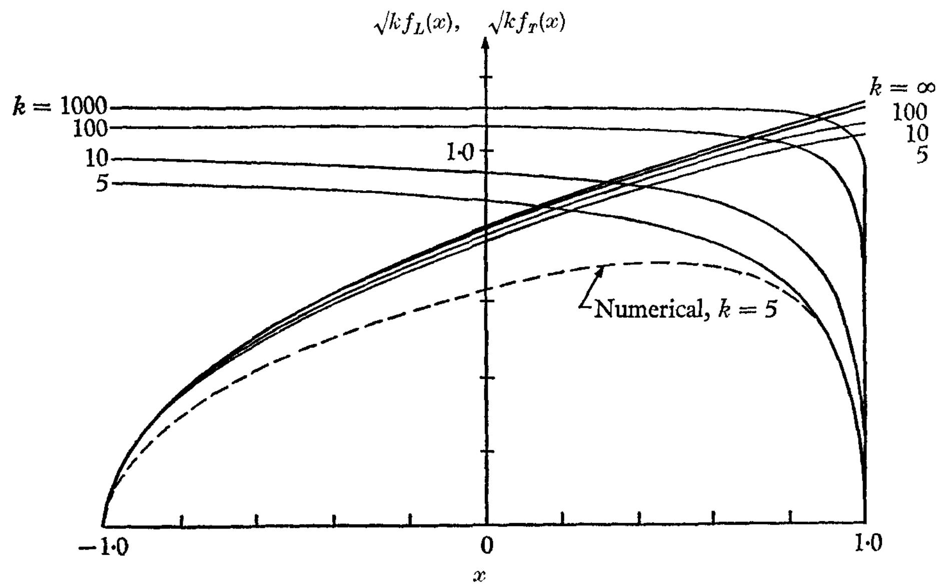

FIGURE 3. Leading- and trailing-edge solutions for the distribution of the magnetic stream-function along the slit in the (x, y)-plane with R_m large.

physical plane is $-q_0 f'(x)$; for the blunt trailing edge $q_0 = O\{\sqrt{(1-x)}\}$ as $x \to 1$. The form of our results for large R_m corroborates the inviscid boundary-layer theory of Sears (1961). One may easily deduce from equation (6) that for $y = O(k^{-1})$ the difference between the magnetic stream-function and the potential-flow stream-function is $O(k^{-\frac{1}{2}})$, except in the neighbourhood of the rear stagnation point. For values of y larger than $O(k^{-\frac{1}{2}})$ this difference becomes exponentially small with y. Furthermore, the upstream influence of the boundary layer is also exponentially small. On the body the current J is given by

$$J(\tilde{x}) = -R_m \frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})}\bigg|_{y=\tilde{y}=0} f'[x(\tilde{x},0)],$$

where $\tilde{x}$, $\tilde{y}$ are Cartesian co-ordinates in the physical plane. Note that for large R_m, $J = O(R_m^{\frac{1}{2}})$ near the body.

Because of the concentration of magnetic field lines in the neighbourhood of the rear stagnation point, we can expect a jet-like distribution of magnetic field lines behind the body. This structure may be computed from equation (6). For large k and with $y = o(x-1)$, the first-order result is

$$\Psi_0^+(x,y) = y + \frac{1}{\pi}\sqrt{\left\{\frac{2(x+1)}{k}\right\}}\int_{z+}^{z^-}\sqrt{(z-z_+)}\,e^{-z}\frac{dz}{z} = y\left[1+O\left(\frac{e^{-ky^2/2x}}{x^{\frac{3}{2}}}\right)\right], \quad (29)$$

where $z_{\pm} = ky^2/2(x \pm y)$. The first term represents magnetic field lines which are coincident with the streamlines, and the second term the slightly increased flux of field lines in the wake. In equation (29) we have taken into account only the distribution $f_0(x)$. The trailing-edge distribution modifies this result by a term is $O(k^{-\frac{1}{3}}\Psi_0^+)$ and has been neglected. This jet-like structure was first pointed out by Tamada (1961*b*).

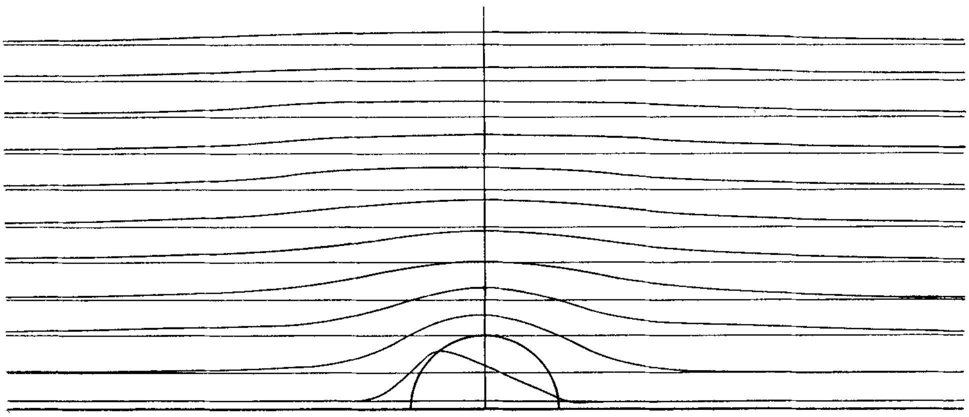

FIGURE 4. Magnetic field for large R_m.

We again give the results for a circular cylinder of unit radius: k is to be interpreted as R_m in order to shrink the slit length to 2, and $\cos^{-1}(-x)$ as the polar angle measured from the forward stagnation point. The magnetic field about the body may be computed by using the composite distribution

$$f(x) = \begin{Bmatrix} f_L(x) & \text{for} & -1 \leqslant x < 1-2(k\pi)^{-\frac{1}{3}} \\ f_T(x) & \text{for} & 1-2(k\pi)^{-\frac{1}{3}} < x \leqslant 1 \end{Bmatrix}.$$

Figure 4 shows the magnetic field computed from equation (6) for $R_m = 100$. Inside the body the field lines are nearly straight. Except in the wake behind the body the magnetic field lines are coincident with the potential flow streamlines at a distance of one or more body radii. In the wake this adjustment occurs more slowly, and requires in the order of ten body radii. The largest deviations occur, of course, in the neighbourhood of the rear stagnation point. Except for this region, computed profiles of the component of the magnetic field parallel to the body H_θ exhibit the expected boundary-layer structure near the body: $H_\theta = q_\theta$ except in the inviscid boundary layer where H_θ falls off to a surface value that is $O(R_m^{-\frac{1}{2}})$. The current at the surface of the body is simply

$$-4R_m \sin^2\theta f'(-\cos\theta).$$

Behind the body the jet-like magnetic field leads to a current distribution in the wake. This wake current is shown at three radial distances r behind the body in figure 5.

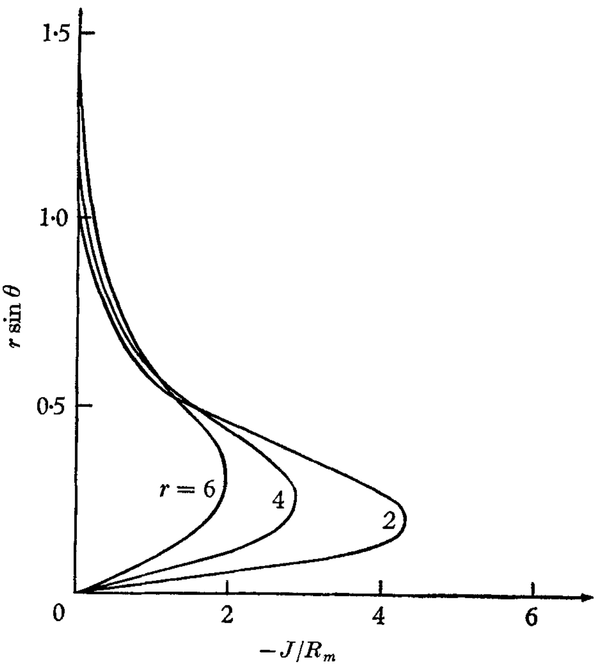

FIGURE 5. Current distribution in the wake.

5. Numerical solution

We may effect a numerical inversion of the singular integral equation by employing our knowledge of the solution determined from the Fredholm inversion for small R_m. Clearly this solution has the form

$$f'(\xi) = \frac{1}{\sqrt{(1-\xi^2)}} \sum_{s=0}^{\infty} A_s \xi^s. \tag{30}$$

The numerical procedure follows that of Stewartson (1964) for a singular integral equation homologous to our equation. The kernel (20) is expanded by using the series expansion of the Bessel functions, with the result

$$\mathscr{K}(u) = \frac{1}{u} + k \sum_{m=0}^{\infty} \frac{k^m u^m}{m!} \left\{ \frac{1}{ku} + \sum_{n=0}^{\infty} (a_n \log 2|u| + b_n) u^n \right\}, \tag{31}$$

where
$$a_{2n} = \frac{-k^{2n}}{2^{2n}(n!)^2}, \quad a_{2n+1} = \frac{k^{2n+1}}{2^{2n+1}n!(n+1)!},$$

$$b_{2n} = a_{2n}(\log \tfrac{1}{4}k - \psi(n+1)), \quad b_{2n+1} = a_{2n+1}\left(\log \tfrac{1}{4}k - \psi(n+1) - \frac{1}{2(n+1)}\right),$$

and $\psi(n)$ is the digamma (psi) function. Series (30) and (31) are then substituted into the integral equation (19), and the integrations performed term by term, which yields a power series in x. This series must be identical to one for all x, and thus we obtain an infinite set of linear algebraic equations for the unknown A_s's

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} A_s M_{st} x^t = 1. \tag{32}$$

For numerical computation the sums in (32) are truncated at a value of s and t, say N, such that subsequent A_s's are negligible. The computation of the elements of the matrix M_{st} is given in the appendix. Since the sums in the computation of M_{st} are also truncated after N terms another, more stringent, requirement on N is that they be convergent.

Because of the singular nature of the kernel of the integral equation (19), a solution is not unique unless we prescribe an additional constraint. For our problem the condition that $f(1)$ should be zero must be satisfied. Thus we replace the last of the equations (32) by the condition

$$\sum_{s=0}^{N} A_s M_{sN}^* = 0,$$

where

$$M_{sN}^* = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{if } s \text{ is odd}, \\ \dfrac{(s-1)(s-3)\dots(1)}{s(s-2)\dots(2)} & \text{if } s \text{ is even.} \end{cases}$$

Once the A_s's have been determined we can carry out the quadrature of $f'(x)$ to obtain the distribution of magnetic-field strength along the slit

$$f(x) = \sum_{m=0}^{N} A_m S_m(x),$$

where

$$S_0 = \sin^{-1}x + \tfrac{1}{2}\pi,$$

$$S_{2m} = -\sqrt{(1-x^2)}\left\{\frac{x^{2m-1}}{2m} + \frac{(2m-1)\,x^{2m-3}}{2m(2m-2)} + \dots + \frac{(2m-1)(2m-3)\dots(3)}{2m(2m-2)\dots(2)}\,x\right\} + \frac{(2m-1)(2m-3)\dots(3)}{2m(2m-2)\dots(2)}\,S_0,$$

and

$$S_{2m+1} = -\sqrt{(1-x^2)}\left\{\frac{x^{2m}}{2m+1} + \frac{2mx^{2m-2}}{(2m+1)(2m-1)} + \dots + \frac{2m(2m-2)\dots(2)}{(2m+1)(2m-1)\dots(1)}\right\}.$$

Numerical computations have been carried out on a Control Data 1604 for values of k ranging from 0·1 to 5·0. The N's required varied from 4 to 60. With $k = 0{\cdot}1$, the values of $f(x)$ for $N = 2$ (three terms) were within one-tenth of one percent of the values for $N = 14$. Although the program could accommodate N's as large as 100, this was not sufficiently large to compute the solution for $k = 10$.

Figure 6 shows the computed values of $f(x)$ for various k's. For values of k less than or equal to 0·5 the results are within 5 % of the values given by our small R_m result (18). Unfortunately, numerical results were unobtainable for values of k such that $(k)^{-\frac{1}{2}} = o(1)$. Nevertheless, the results for $k = 5$, which are also plotted in figure 3, indicate that our large k theory is a satisfactory one. With certain modifications the program could be extended beyond its present limit of $N = 100$.

With values of k of 10 or larger the elements of the matrix would vary to such an extent that double precision might be required to effect an accurate inversion. For this reason we have not attempted such modifications.

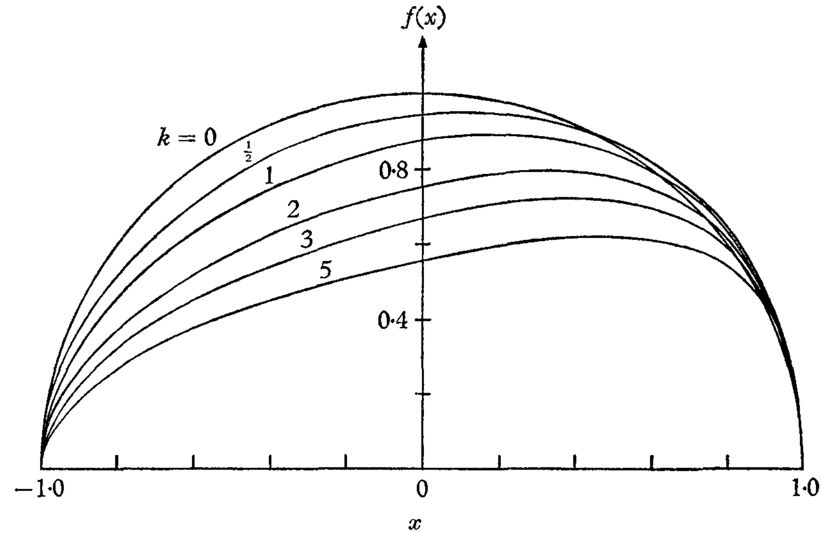

FIGURE 6. Computed distribution of the magnetic stream-function along the slit in the (x, y)-plane.

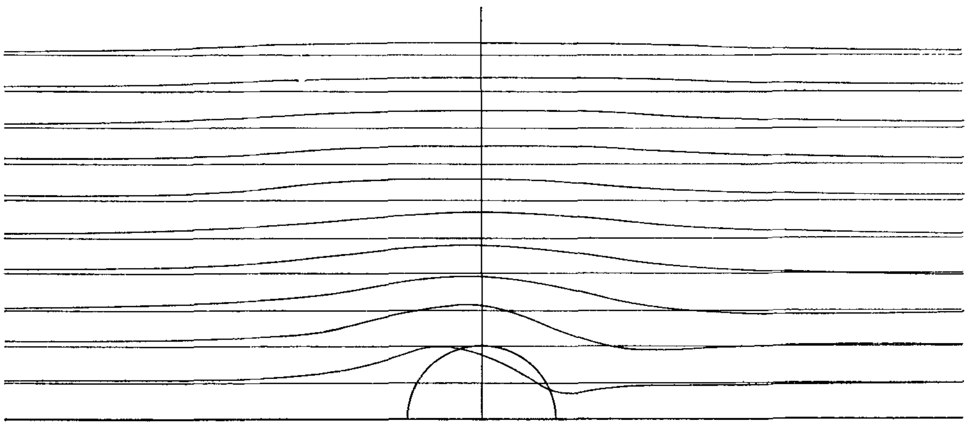

FIGURE 7. Magnetic field for moderate R_m.

One notable result is that for distances of several radii from the body the computed magnetic field for $R_m = 5$ is indistinguishable from our analytical results for $R_m = 100$, provided we are not in the wake behind the body. Since at these distances from the body our analytical solution for the magnetic stream-function with $R_m = 100$ coincides with the potential flow stream-function, we can conclude that even at moderate values of R_m the magnetic field lines follow the streamlines except in the neighbourhood of the body and in the wake behind the body.

The authors are indebted to Prof. C. K. Chu of Columbia University for his valuable suggestions, to Mr Y. Sone of Kyoto University for his help with the analysis, to Mrs J. W. Jack and Mrs R. Bernabei who carried out the numerical programming, and to Mr B. P. Leonard for his helpful discussions. This study was supported in part by the U.S. Air Force Office of Scientific Research.

Appendix

The elements of the matrix M_{st} occurring in equation (32) are simply a bilinear combination of the coefficients of the expansion of the kernel and the coefficients of the powers of x that result from performing the integration term by term. Stewartson (1964) has tabulated these integrals in connexion with a numerical solution of a similar integral equation. The integrals which arise are:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^n}{(x-\xi)}\frac{d\xi}{\sqrt{(1-\xi^2)}} = \sum_{r=0}^{n-1} P_{n-r-1}x^r,$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^n(x-\xi)^m}{\sqrt{(1-\xi^2)}}\,d\xi = \sum_{r=0}^{m}(-1)^{m-r+1}\binom{m}{r}P_{n+m-r}x^r \equiv \sum_{r=0}^{m} b_r(m,n)\,x^r,$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^n(x-\xi)^m}{\sqrt{(1-\xi^2)}}\log 2|x-\xi|\,d\xi = \sum_{r=0}^{m}(-1)^{m-r+1}\binom{m}{r}\left[Q_{n+m-r}+P_{m+n-r}\sum_{t=0}^{r-1}\frac{1}{m-t}\right]x^r$$
$$+\sum_{r=m+1}^{m+n}\frac{m!(r-m-1)!}{r!}P_{n+m-r}x^r \equiv \sum_{r=0}^{m+n} c_r(m,n)\,x^r.$$

Here $\binom{m}{r}$ are the binomial coefficients $m, n = 0, 1, 2, \ldots$,

$$P_{2m} = -\frac{(2m)!}{2^{2m}(n!)^2} \quad\text{and}\quad Q_{2m} = -P_{2m}\sum_{r=1}^{2m}\frac{(-1)^r}{r}.$$

Undefined values of the symbols are zero. The coefficients $c_r(m,n)$ and $b_r(m,n)$ are defined, following Stewartson, to simplify the expression for the matrix. Our results for these integrals agree with those of Stewartson except for a minor error in his values for the Q_{2m}'s.

Finally, we may express the elements of the matrix as

$$M_{st} = +2P_{s-t-1}+k\sum_{m=0}^{\infty}\left[c_t(m,s)\sum_{r=0}^{m}\frac{k^{m-r}}{(m-r)!}a_r+b_t(m,s)\left(\frac{k^m}{(m+1)!}+\sum_{r=0}^{m}b_r\frac{k^{m-r}}{(m-r)!}\right)\right],$$

where the a_r's and b_r's are obtained from the series expansion of the kernel (31). The sum on m is, of course, truncated at the chosen value of N.

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